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# Chiral Invariant Renormalization of the Pion–Nucleon Interaction*

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**Abstract**

The leading divergences of the generating functional for Green functions of quark currents between one–nucleon states are calculated with heat kernel techniques. The results allow for a chiral invariant renormalization of all two–nucleon Green functions of the pion–nucleon system to $O(p^3)$ in the low–energy expansion.

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**1.** The modern treatment of the pion–nucleon interaction as an effective field theory of the standard model was pioneered by Gasser, Sainio and Švarc [1] who applied the methods of chiral perturbation theory (CHPT) to the $\pi N$ system. However, the chiral expansion with baryons is hampered by the presence of the nucleon mass, which stays finite in the chiral limit. It was then shown by Jenkins and Manohar [2] that the methods of heavy quark effective theory [3] allow for a systematic low–energy expansion of baryonic Green functions in complete analogy to the meson sector.

The purpose of this letter is to perform the complete renormalization of Green functions of quark currents between one–nucleon states to $O(p^3)$ in the chiral expansion. This requires a chiral invariant calculation of the divergent part of the corresponding generating functional at the one–loop level. Many specific Green functions of the $\pi N$ system have already been analyzed to one–loop accuracy [1, 4, 5, 6]. However, only the full divergence structure of $O(p^3)$ presented here permits a complete renormalization of all Green functions: nucleon form factors, $\pi N \to \pi \ldots \pi N$, $\gamma^* N \to \pi \ldots \pi N$, $W^* N \to \pi \ldots \pi N$, etc. Results will be given for two light flavours only. All details and the extension to chiral $SU(3)$ will be presented elsewhere.

We start from QCD with two light flavours $u, d$ coupled to external hermitian fields [7]:

$$\mathcal{L} = \mathcal{L}^0_{\rm QCD} + \bar{q}\gamma^\mu \left( \mathcal{V}_\mu + \frac{1}{3}\mathcal{V}^s_\mu + \gamma_5 \mathcal{A}_\mu \right) q - \bar{q}(S - i\gamma_5 P)q, \qquad q = \begin{pmatrix} u \\ d \end{pmatrix}. \tag{1}$$

$S, P$ are general 2–dimensional matrix fields, the isotriplet vector and axial–vector fields $\mathcal{V}_\mu, \mathcal{A}_\mu$ are traceless and the isosinglet vector field $\mathcal{V}^s_\mu$ is included to generate the electromagnetic current. At the effective level of pions and nucleons, $\mathcal{V}^s$ couples directly only to nucleons since the pions have zero baryon number.

Explicit chiral symmetry breaking is implemented by setting $S = \mathcal{M} = \mathrm{diag}\,(m_u, m_d)$. The chiral group $G = SU(2)_L \times SU(2)_R$ breaks spontaneously to $SU(2)_V$ (isospin). It is realized non–linearly [8] on the Goldstone pion fields $\phi$:

$$\begin{aligned} \xi_L(\phi) &\stackrel{g}{\to} g_L \xi_L(\phi) h(g,\phi)^{-1}, \qquad g = (g_L, g_R) \in G \\ \xi_R(\phi) &\stackrel{g}{\to} g_R \xi_R(\phi) h(g,\phi)^{-1}, \end{aligned} \tag{2}$$

where $\xi_L, \xi_R$ are elements of the chiral coset space $SU(2)_L \times SU(2)_R / SU(2)_V$ and the compensator field $h(g,\phi)$ is in $SU(2)_V$. The more familiar quantity $U(\phi) = \xi_R(\phi)\xi_L(\phi)^\dagger$ transforms linearly under $G$. In the standard "gauge" with $\xi_R = \xi_L^\dagger =: u$ we have $U = u^2$.

The nucleon doublet $\Psi$ transforms as

$$\Psi = \begin{pmatrix} p \\ n \end{pmatrix} \stackrel{g}{\to} \Psi' = h(g,\phi)\Psi \tag{3}$$

under chiral transformations. The local nature of this transformation requires a connection

$$\Gamma_\mu = \frac{1}{2}\{\xi_R^\dagger(\partial_\mu - i r_\mu)\xi_R + \xi_L^\dagger(\partial_\mu - i\ell_\mu)\xi_L\} \tag{4}$$

in the presence of external gauge fields

$$r_\mu = \mathcal{V}_\mu + \mathcal{A}_\mu, \qquad \ell_\mu = \mathcal{V}_\mu - \mathcal{A}_\mu \tag{5}$$

to define a covariant derivative

$$\nabla_\mu \Psi = (\partial_\mu + \Gamma_\mu - i\mathcal{V}^s_\mu)\Psi. \tag{6}$$

The effective chiral Lagrangian for Green functions with at most two nucleons is [1, 7]

$$\begin{aligned}
\mathcal{L}_{\text{eff}} &= \mathcal{L}_M + \mathcal{L}_{MB} && (7)\\
\mathcal{L}_M &= \mathcal{L}_2 + \mathcal{L}_4 + \ldots \\
\mathcal{L}_2 &= \frac{F^2}{4}\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U\rangle = \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+\rangle && (8)\\
& \quad u_\mu = i\{\xi_R^\dagger(\partial_\mu - ir_\mu)\xi_R - \xi_L^\dagger(\partial_\mu - i\ell_\mu)\xi_L\} \\
& \quad \chi = 2B_0(S+iP), \qquad \chi_\pm = u^\dagger\chi u^\dagger \pm u\chi^\dagger u \\
\mathcal{L}_{MB} &= \mathcal{L}^{(1)}_{\pi N} + \mathcal{L}^{(2)}_{\pi N} + \mathcal{L}^{(3)}_{\pi N} + \ldots \\
\mathcal{L}^{(1)}_{\pi N} &= \bar\Psi(i\not\nabla - m + \frac{g_A}{2}\not u\gamma_5)\Psi && (9)
\end{aligned}$$

where $m, g_A$ are the nucleon mass and the neutron decay constant in the chiral limit and $\langle\ldots\rangle$ stands for the trace in flavour space.

The generating functional of Green functions $Z[j,\eta,\bar\eta]$ is defined [1] by the path integral

$$e^{iZ[j,\eta,\bar\eta]} = N\int [dud\Psi d\bar\Psi]\exp[i\{S_M + S_{MB} + \int d^4x(\bar\eta\Psi + \bar\Psi\eta)\}]. \tag{10}$$

The action $S_M + S_{MB}$ corresponds to the effective Lagrangian (7), the external fields $\mathcal{V}, \mathcal{A}, S, P$ are denoted collectively as $j$ and $\eta, \bar\eta$ are fermionic sources.

**2.** Heavy baryon CHPT [2] can be viewed as a clever choice of variables for performing the fermionic path integral in (10). By shifting the dependence on the nucleon mass $m$ from the nucleon propagator to the vertices of the effective Lagrangian, the integration over the new fermionic variables produces a systematic low–energy expansion.

In terms of the velocity–dependent fields $N_v, H_v$ defined as

$$\begin{aligned}
N_v(x) &= \exp[imv\cdot x]P_v^+\Psi(x) && (11)\\
H_v(x) &= \exp[imv\cdot x]P_v^-\Psi(x) \\
P_v^\pm &= \frac{1}{2}(1\pm\not v)\ , \qquad v^2 = 1\ ,
\end{aligned}$$

the pion–nucleon action $S_{MB}$ takes the form

$$\begin{aligned}
S_{MB} &= \int d^4x\{\bar N_v A N_v + \bar H_v B N_v + \bar N_v\gamma^0 B^\dagger\gamma^0 H_v - \bar H_v C H_v\} && (12)\\
A &= iv\cdot\nabla + g_A S\cdot u + A_{(2)} + A_{(3)} + \ldots \\
B &= i\not\nabla^\perp - \frac{g_A}{2}v\cdot u\gamma_5 + B_{(2)} + B_{(3)} + \ldots \\
C &= 2m + iv\cdot\nabla + g_A S\cdot u + C_{(2)} + C_{(3)} + \ldots \\
& \nabla^\perp_\mu = \nabla_\mu - v_\mu v\cdot\nabla, \qquad [\not v, A] = [\not v, C] = 0, \qquad \{\not v, B\} = 0.
\end{aligned}$$

In $A$ and $C$, the only dependence on Dirac matrices is through the spin matrix

$$S^\mu = \frac{i}{2}\gamma_5\sigma^{\mu\nu}v_\nu, \qquad S\cdot v = 0, \qquad S^2 = -\frac{3}{4}\mathbf{1}. \tag{13}$$

Rewriting also the source term in (10) in terms of $N_v, H_v$ with corresponding sources

$$\rho_v = e^{imv\cdot x}P_v^+\eta, \qquad R_v = e^{imv\cdot x}P_v^-\eta\ , \tag{14}$$

one can now integrate out the "heavy" components $H_v$ to obtain a non–local action in the "light" fields $N_v$ [9, 10, 11]. At this point, the crucial approximation of heavy baryon CHPT is made: the action is written as a series of local actions with increasing chiral dimensions by expanding $C^{-1}$ in a power series in $1/m$:

$$C^{-1} = \frac{1}{2m} - \frac{iv\cdot\nabla + g_A S\cdot u}{(2m)^2} + O(p^2)\ . \tag{15}$$

With this approximation, the integration over $N_v$ reduces again to completing a square (the fermion determinant is trivial to any finite order in $1/m$) with the final result

$$e^{iZ[j,\eta,\bar\eta]} = N\int[du]e^{i(S_M + Z_{MB}[u,j,\rho_v,R_v])} \tag{16}$$

where

$$\begin{aligned} Z_{MB}[u,j,\rho_v,R_v] = -\int d^4x\{&\bar\rho_v(A+\gamma^0B^\dagger\gamma^0C^{-1}B)^{-1}\rho_v \\ &+ \bar R_vC^{-1}B(A+\gamma^0B^\dagger\gamma^0C^{-1}B)^{-1}\rho_v + \bar\rho_v(A+\gamma^0B^\dagger\gamma^0C^{-1}B)^{-1}\gamma^0B^\dagger\gamma^0C^{-1}R_v \\ &+ \bar R_vC^{-1}B(A+\gamma^0B^\dagger\gamma^0C^{-1}B)^{-1}\gamma^0B^\dagger\gamma^0C^{-1}R_v - \bar R_vC^{-1}R_v\}. \end{aligned} \tag{17}$$

The functional integral (10) has been reduced to the mesonic integral (16). From here on, the standard procedure of CHPT [7] can be applied: the action in the functional integral (16) is expanded around the classical solution $u_{\rm cl}[j]$ of the lowest–order equation of motion. Integration over the quantum fluctuations gives rise to a well–behaved low–energy expansion for $Z[j,\eta,\bar\eta]$ like in the meson sector.

For $u = u_{\rm cl}$, we can read off the tree–level nucleon propagator from the right–hand side of Eq. (17) (see Fig. 1 of Ref. [1] for an artistic impression of this object):

$$Z_{MB}^{\rm tree}[j,\eta,\bar\eta] = Z_{MB}[u_{\rm cl}[j],j,\rho_v,R_v]. \tag{18}$$

By construction, $Z_{MB}^{\rm tree}$ is independent of the arbitrary vector $v$. It is instructive to check this independence for the free propagator corresponding to [1]

$$A = iv\cdot\partial, \qquad B = i\ \partial\!\!\!/^{\perp}, \qquad C = iv\cdot\partial + 2m. \tag{19}$$

Inserting these operators into the right–hand side of Eq. (17) and reexpressing $\rho_v, R_v$ in terms of the original fermionic sources, one finds indeed the free massive fermion propagator

$$Z_{MB}^{\rm tree}[0,\eta,\bar\eta] = \int d^4x\bar\eta(x)(i\ \partial\!\!\!/ + m)(\Box + m^2)^{-1}\eta(x). \tag{20}$$

On the other hand, any given order in the chiral expansion of $Z_{MB}[j,\eta,\bar\eta]$ will in general not be independent of $v$ because a change in $v$ involves different chiral orders (reparametrization invariance [12]). This is the price one has to pay for a systematic low–energy expansion.

[1] $C^{-1}$ is not expanded in this case.

**3.** We now turn to the calculation of $Z_{MB}[j,\eta,\bar\eta]$ up to and including $O(p^3)$. Since we only consider nucleons (rather than antinucleons), we can drop the sources $R_v$ to the order considered. To $O(p^2)$, $Z_{MB}[j,\eta,\bar\eta]$ is a pure tree–level functional. The relevant pion–nucleon Lagrangian is [2, 10]

$$\bar N_v(A+\gamma^0 B^\dagger\gamma^0 C^{-1}B)N_v = \bar N_v\left(A_{(1)}+A_{(2)}+\frac{1}{2m}\gamma^0 B^\dagger_{(1)}\gamma^0 B_{(1)}\right)N_v + O(p^3) \qquad (21)$$

$$A_{(1)} = iv\cdot\nabla + g_A S\cdot u \qquad B_{(1)} = i\,\not\nabla^\perp - \frac{g_A}{2}v\cdot u\gamma_5$$

$$\begin{aligned} A_{(2)} &= C_1\langle u\cdot u\rangle + C_2\langle (v\cdot u)^2\rangle + C_3\chi_+ + C_4\langle\chi_+\rangle \\ &\quad + \varepsilon^{\mu\nu\rho\sigma}v_\rho S_\sigma\{C_5 u_\mu u_\nu + C_6 f_{+\mu\nu} + C_7(\partial_\mu\mathcal{V}^s_\nu - \partial_\nu\mathcal{V}^s_\mu)\} \\ f^{\mu\nu}_\pm &= uF^{\mu\nu}_L u^\dagger \pm u^\dagger F^{\mu\nu}_R u, \end{aligned} \qquad (22)$$

where $F_{L,R}$ are the field strengths associated with the isotriplet gauge fields $\ell, r$ in (5). Since the loop contributions set in at $O(p^3)$ only, neither $g_A$ nor the coupling constants $C_1,\ldots,C_7$ of $O(p^2)$ will have to be renormalized.

To calculate the loop functional of $O(p^3)$, we expand

$$\mathcal{L}_2 + \mathcal{L}_4 - \bar\rho_v A^{-1}_{(1)}\rho_v \qquad (23)$$

in the functional integral (16) around the classical solution $u_{\rm cl}[j]$. A convenient choice of fluctuation variables $\xi$ is given by [7]

$$\ell_R(\phi) = u(\phi_{\rm cl})e^{i\xi(\phi)/2}, \qquad \ell_L(\phi) = u^\dagger(\phi_{\rm cl})e^{-i\xi(\phi)/2}, \qquad \xi^\dagger=\xi, \qquad \langle\xi\rangle = 0, \qquad \xi(\phi_{\rm cl})=0. \qquad (24)$$

Following Gasser, Sainio and Švarc [1], we expand $\mathcal{L}_2$ to $O(\xi^3)$, $\mathcal{L}_4$ to $O(\xi)$ and $A^{(1)}$ to $O(\xi^2)$ to arrive at the diagrams of Figs. 1,2. The sum of the reducible diagrams in Fig. 2 is finite and scale independent [7]. For the irreducible diagrams of Fig. 1, we need

$$\begin{aligned} A_{(1)} &= iv\cdot\nabla + g_A S\cdot u \\ &= iv\cdot\nabla_{\rm cl} + g_A S\cdot u_{\rm cl} + \frac{i}{4}[v\cdot u_{\rm cl},\xi] - g_A S\cdot\nabla_{\rm cl}\xi \\ &\quad + \frac{i}{8}\xi v\cdot \overleftrightarrow{\nabla}_{\rm cl}\,\xi + \frac{g_A}{8}[\xi,[S\cdot u_{\rm cl},\xi]] + O(\xi^3) \\ \nabla^\mu_{\rm cl}\xi &:= \partial^\mu\xi + [\Gamma^\mu_{\rm cl},\xi]. \end{aligned} \qquad (25)$$

From now on, the index cl will be dropped. All mesonic quantities are to be taken at the classical solution of the lowest–order equation of motion.

The diagrams of Fig. 1 correspond to the following generating functional:

$$Z_{\rm irr}[j,\rho_v] = \int d^4x d^4x' d^4y d^4y' \bar\rho_v(x) A^{-1}_{(1)}(x,y)[\Sigma_1(y,y')\delta^4(y-y') + \Sigma_2(y,y')]\cdot A^{-1}_{(1)}(y',x')\rho_v(x') \qquad (26)$$

where $A_{(1)}^{-1}$ is the propagator for $N_v$ in the presence of external fields. The self–energy functionals $\Sigma_1, \Sigma_2$ are given by

$$\begin{aligned}
\Sigma_1(y,y') &= \frac{1}{8F^2}\{i\tau_i[G_{ij}(y,y')v\cdot \overleftarrow{d'}_{jk} - v\cdot d_{ij}G_{jk}(y,y')]\tau_k + g_A[\tau_i,[S\cdot u,\tau_j]]G_{ij}(y,y')\} && (27)\\
\Sigma_2(y,y') &= -\frac{2}{F^2}V_i(y)G_{ij}(y,y')A_{(1)}^{-1}(y,y')V_j(y') && (28)\\
V_i &= \frac{i}{4\sqrt{2}}[v\cdot u,\tau_i] - \frac{g_A}{\sqrt{2}}\tau_j S\cdot d_{ji}.
\end{aligned}$$

The differential operator $d_{ij}^\mu$ in 3–dimensional tangent space is related to the previously defined covariant derivative as

$$\nabla^\mu\xi = \frac{1}{\sqrt{2}}\tau_i d_{ij}^\mu\xi_j, \qquad \xi = \frac{1}{\sqrt{2}}\tau_i\xi_i, \qquad d_{ij}^\mu = \delta_{ij}\partial^\mu + \gamma_{ij}^\mu, \qquad \gamma_{ij}^\mu = -\frac{1}{2}\langle\Gamma^\mu[\tau_i,\tau_j]\rangle. \qquad (29)$$

It acts on the meson propagator $G_{ij}$:

$$G = (d_\mu d^\mu + \sigma)^{-1}, \qquad \sigma_{ij} = \frac{1}{8}\langle[u_\mu,\tau_i][\tau_j,u^\mu] + \chi_+\{\tau_i,\tau_j\}\rangle. \qquad (30)$$

**4.** The self–energy functionals $\Sigma_1(y,y')$, $\Sigma_2(y,y')$ are divergent for $y'\to y$. In order to extract the divergences in a chiral invariant manner, we use the heat kernel representation of propagators (see Ref. [13] for a review) in $d$–dimensional Euclidean space. The divergences will appear as simple poles in $\varepsilon = \frac{1}{2}(4-d)$. The corresponding residua are local polynomials in the fields of $O(p^3)$ and can easily be transformed back to Minkowski space.

In Euclidean space with $d$ dimensions, the inverse of the elliptic second–order differential operator

$$D_2 = -d_\mu d_\mu + \sigma \qquad (31)$$

can be constructed as an integral

$$G(x,y) = D_2^{-1}(x,y) = \int_0^\infty d\lambda G(x,y;\lambda) \qquad (32)$$

over the heat kernel $G(x,y;\lambda)$ with its asymptotic expansion for $\lambda\to 0_+$ :

$$\begin{aligned}
&G(x,y;\lambda) = \frac{1}{(4\pi\lambda)^{d/2}}e^{-(x-y)^2/4\lambda}\sum_n a_n(x,y)\lambda^n\\
&\left(\frac{\partial}{\partial\lambda} + D_2\right)G(x,y;\lambda) = 0, \qquad G(x,y;0) = \delta^d(x-y). \qquad (33)
\end{aligned}$$

The differential equation for $G(x,y;\lambda)$ yields recursion relations for the Seeley–DeWitt coefficients $a_n$, which are matrices in tangent space of $O(p^{2n})$. Since the divergences of $\Sigma_1, \Sigma_2$ are $O(p^3)$, we only need the following derivatives of $a_0, a_1$ in the coincidence limit $x\to y$, which is

to be understood in the list below:

$$
\begin{aligned}
& a_0 = \mathbf{1} && \\
& d_\mu a_0 = a_0 \overleftarrow{d}_\mu = 0 && d_\mu a_n = \lim_{x\to y}[\partial^x_\mu + \gamma_\mu(x)]a_n(x,y) \\
& d_\mu d_\nu a_0 = d_\mu a_0 \overleftarrow{d}_\nu = \frac{1}{2}\gamma_{\mu\nu} && a_n \overleftarrow{d}_\mu = \lim_{x\to y}[\partial^y_\mu a_n(x,y) - a_n(x,y)\gamma_\mu(y)] \\
& d_\lambda d_\mu d_\nu a_0 = 2 d_\lambda d_\mu a_0 \overleftarrow{d}_\nu = \frac{1}{3}(d_\lambda \gamma_{\mu\nu} + d_\mu \gamma_{\lambda\nu}) && \gamma_{\mu\nu} = \partial_\mu\gamma_\nu - \partial_\nu\gamma_\mu + [\gamma_\mu, \gamma_\nu] \\
& a_1 = -\sigma && d_\lambda\gamma_{\mu\nu} = \partial_\lambda\gamma_{\mu\nu} + [\gamma_\lambda, \gamma_{\mu\nu}] \\
& d_\mu a_1 = \frac{1}{6} d_\nu \gamma_{\mu\nu} - \frac{1}{2} d_\mu \sigma && \\
& a_1 \overleftarrow{d}_\mu = -\frac{1}{6} d_\nu \gamma_{\mu\nu} - \frac{1}{2} d_\mu \sigma. &&
\end{aligned} \tag{34}
$$

In $d$ dimensions, we have [13, 14]

$$
\begin{aligned}
& G(x,x) = \frac{a_1(x,x)}{(4\pi)^2\varepsilon}, \qquad \varepsilon = \frac{4-d}{2} \\
& \lim_{x\to y} \partial^x_\mu(\partial^y_\mu) G(x,y) = \frac{1}{(4\pi)^2\varepsilon} \lim_{x\to y} \partial^x_\mu(\partial^y_\mu) a_1(x,y).
\end{aligned} \tag{35}
$$

Inserting the coincidence relations for $a_1$ into $\Sigma_1$ in (27), performing the summation over the tangent space indices $i, j, k$ and transforming back to Minkowski space, one obtains the divergent part of $\Sigma_1$, which is manifestly of $O(p^3)$:

$$
\Sigma_1^{\rm div}(y,y) = \frac{1}{(4\pi F)^2\varepsilon}\widehat{\Sigma}_1(y) \tag{36}
$$

$$
\begin{aligned}
\widehat{\Sigma}_1(y) &= -\frac{i}{6}\nabla^\mu\Gamma_{\mu\nu}v^\nu + \frac{g_A}{8}\left(\{S\cdot u, u\cdot u + \chi_+\} + S\cdot u\langle u\cdot u + \chi_+\rangle\right. \\
&\quad \left. + 2u_\mu\langle u^\mu S\cdot u\rangle - \langle S\cdot u u\cdot u\rangle - \langle S\cdot u\chi_+\rangle\right) \\
\Gamma_{\mu\nu} &= \partial_\mu\Gamma_\nu - \partial_\nu\Gamma_\mu + [\Gamma_\mu, \Gamma_\nu].
\end{aligned} \tag{37}
$$

**5.** The divergences of $\Sigma_2(y,y')$ are due to the singular behaviour for $y \to y'$ of the product of (derivatives of) propagators

$$
G_{ij}(y,y')A^{-1}_{(1)}(y,y') \; . \tag{38}
$$

In accordance with locality, the divergences are of the generic form (with an overall chiral dimension 3)

$$
\frac{1}{(4\pi F)^2\varepsilon}F(y,y')D(y;v)\delta^4(y-y') \tag{39}
$$

with a field monomial $F(y,y')$ and a differential operator $D(y;v)$ of at most third order.

The missing ingredient is the heat kernel representation for the nucleon propagator $A^{-1}_{(1)}$. Below, only results needed for the present analysis are collected. However, the following representation will clearly also be useful in heavy quark effective theory.

Since $A_{(1)}$ is not an elliptic differential operator, we write

$$A_{(1)}^{-1} = i\mathcal{D}^\dagger(\mathcal{D}\mathcal{D}^\dagger)^{-1}, \qquad \mathcal{D} = iA_{(1)} \tag{40}$$

and set up a heat kernel representation for the inverse of the positive definite hermitian operator $\Delta = \mathcal{D}\mathcal{D}^\dagger$, where

$$\begin{aligned} \mathcal{D} &= -v\cdot\nabla - ig_A S\cdot u + \eta v^2, \qquad \eta \to 0_+ \\ \Delta &= \mathcal{D}\mathcal{D}^\dagger = -(v\cdot\nabla)^2 + R + \eta^2 \\ R &= ig_A(v\cdot\nabla S\cdot u) - g_A^2(S\cdot u)^2. \end{aligned} \tag{41}$$

To facilitate the transformation back to Minkowski space, it is preferable to keep track of the dependence on $v^2$ in Euclidean space. To get a well–defined representation for $\Delta^{-1}$ without infrared divergences, one must keep $\eta \neq 0$ in intermediate steps. The heat kernel representation for $\Delta^{-1}$ can be motivated by remembering that $\Delta$ is essentially a one–dimensional differential operator in the direction of $v$:

$$\Delta^{-1}(x,y) = \int_0^\infty d\tau\; E(x,y;\tau) \tag{42}$$

$$\left(\frac{\partial}{\partial\tau} + \Delta\right) E(x,y;\tau) = 0, \qquad E(x,y;0) = \delta^d(x-y)$$

$$E(x,y;\tau) = \frac{1}{\sqrt{4\pi\tau}} \exp\left[-\frac{[v\cdot(x-y)]^2}{4\tau} - \eta^2\tau\right] g(x-y;v) \sum_n b_n(x,y)\tau^n$$

$$g(x;v) = \int \frac{d^dk}{(2\pi)^{d-1}} \delta(k\cdot v) e^{-ik\cdot x}, \qquad v\cdot\partial g = 0.$$

The differential equation for $E$ produces again recursion relations for the coefficients $b_n$, which are matrices in 2–dimensional flavour space. The following values of the $b_n(x,x)$ are needed for the present analysis:

$$\begin{aligned} b_0 &= \mathbf{1}, \qquad (v\cdot\nabla)^n b_0 = 0 \\ b_1 &= -R, \qquad v\cdot\nabla b_1 = -\frac{1}{2} v\cdot\nabla R. \end{aligned} \tag{43}$$

In addition to the Seeley–DeWitt coefficients $a_n, b_n$ and their derivatives, we need the coincidence limits of (derivatives of) products of the meson and nucleon propagator functions

$$G_n(x) = \int_0^\infty d\lambda \frac{\lambda^n \exp[-\frac{x^2}{4\lambda}]}{(4\pi\lambda)^{d/2}}, \qquad E_m(x;v) = g(x;v)\int_0^\infty d\tau \frac{\tau^m \exp[-\frac{(v\cdot x)^2}{4\tau} - \eta^2\tau]}{\sqrt{4\pi\tau}}. \tag{44}$$

The corresponding singularities can best be extracted in Euclidean $d$-dimensional Fourier space [14]. The following list contains all singular products appearing in $\Sigma_2$:

$$G_0(x)E_0(x;v) \sim -\frac{2}{(4\pi)^2 v^2\varepsilon}\delta^d(x) \qquad \varepsilon = \frac{1}{2}(4-d) \tag{45}$$

$$G_0(x) v\cdot\partial E_0(x;v) \sim \frac{2}{(4\pi)^2 v^2\varepsilon} v\cdot\partial\delta^d(x) \tag{46}$$

$$S_\mu S_\nu \partial_\mu \partial_\nu G_0(x) E_0(x;v) \sim \frac{2}{(4\pi)^2\varepsilon} S_\mu S_\nu \delta_{\mu\nu} (v\cdot\partial)^2 \delta^d(x) \tag{47}$$

$$S_\mu S_\nu \partial_\mu \partial_\nu G_0(x) v\cdot\partial E_0(x;v) \sim -\frac{2}{3(4\pi)^2\varepsilon} S_\mu S_\nu \delta_{\mu\nu} (v\cdot\partial)^3 \delta^d(x) \tag{48}$$

$$S_\mu S_\nu \partial_\mu \partial_\nu G_0(x) E_1(x;v) \sim -\frac{2}{3(4\pi)^2\varepsilon} S_\mu S_\nu \delta_{\mu\nu} \delta^d(x) \tag{49}$$

$$S_\mu S_\nu \partial_\mu \partial_\nu G_0(x) v\cdot\partial E_1(x;v) \sim \frac{2}{(4\pi)^2\varepsilon} S_\mu S_\nu \delta_{\mu\nu} v\cdot\partial \delta^d(x) \tag{50}$$

$$S_\mu S_\nu \partial_\mu \partial_\nu G_1(x) E_0(x;v) \sim \frac{1}{(4\pi)^2 v^2\varepsilon} S_\mu S_\nu \delta_{\mu\nu} \delta^d(x) \tag{51}$$

$$S_\mu S_\nu \partial_\mu \partial_\nu G_1(x) v\cdot\partial E_0(x;v) \sim -\frac{1}{(4\pi)^2 v^2\varepsilon} S_\mu S_\nu \delta_{\mu\nu} v\cdot\partial\delta^d(x). \tag{52}$$

The actual calculation of the divergent part of $\Sigma_2$ is rather tedious. The final result can be expressed in terms of a local functional $\widehat{\Sigma}_2(y)$ defined in analogy with (36) :

$$\Sigma_2^{\rm div}(y,y') = \frac{1}{(4\pi F)^2\varepsilon}\widehat{\Sigma}_2(y)\delta^d(y-y'). \tag{53}$$

Transforming back to Minkowski space, one finds

$$\begin{aligned}
\widehat{\Sigma}_2(y) = {} & i\left\{\frac{1}{4}[2(v\cdot u)^2 + \langle (v\cdot u)^2\rangle]v\cdot\nabla + \frac{1}{2}v\cdot u(v\cdot\nabla v\cdot u) + \frac{1}{4}\langle v\cdot u(v\cdot\nabla v\cdot u)\rangle\right\} \\
& + g_A\left\{-\frac{1}{2}v\cdot u\langle S\cdot u v\cdot u\rangle + \frac{1}{4}\langle S\cdot u(v\cdot u)^2\rangle - S^\mu v^\nu[\Gamma_{\mu\nu}, v\cdot u]\right\} \\
& + ig_A^2\left\{-\frac{3}{2}(v\cdot\nabla)^3 - \frac{5}{6}\nabla^\mu\Gamma_{\mu\nu}v^\nu + i\varepsilon^{\mu\nu\rho\sigma}v_\rho S_\sigma[2\Gamma_{\mu\nu}v\cdot\nabla + (v\cdot\nabla\Gamma_{\mu\nu})]\right. \\
& - \frac{3}{8}\langle u_\mu(v\nabla u^\mu)\rangle - \frac{3}{8}(v\cdot\nabla u\cdot u) + -\frac{9}{32}\langle v\cdot\partial\chi_+\rangle \\
& \left. - \frac{3}{8}[\langle u\cdot u\rangle + 2u\cdot u + \frac{3}{2}\langle\chi_+\rangle]v\cdot\nabla\right\} \\
& + g_A^3\left\{-\frac{1}{2}S\cdot u(v\cdot\nabla)^2 - 2S^\mu\langle S\cdot u\Gamma_{\mu\nu}\rangle S^\nu - \frac{1}{2}(v\cdot\nabla S\cdot u)v\cdot\nabla\right. \\
& - \frac{1}{6}((v\cdot\nabla)^2 S\cdot u) - \frac{1}{4}u_\mu\langle u^\mu S\cdot u\rangle + \frac{1}{8}\langle S\cdot u(u\cdot u + \chi_+)\rangle \\
& \left. - \frac{1}{8}\{\chi_+, S\cdot u\} + \frac{1}{16}S\cdot u\langle\chi_+\rangle\right\} \\
& + ig_A^4 S_\mu\left\{[2(S\cdot u)^2 - 4\langle(S\cdot u)^2\rangle]v\cdot\nabla + \frac{2}{3}(v\cdot\nabla S\cdot u)S\cdot u\right. \\
& \left. + \frac{4}{3}S\cdot u(v\cdot\nabla S\cdot u) - 4\langle S\cdot u(v\cdot\nabla S\cdot u)\rangle\right\}S^\mu \\
& + g_A^5 S_\mu\left\{\frac{2}{3}(S\cdot u)^3 - \frac{4}{3}\langle(S\cdot u)^3\rangle\right\}S^\mu.
\end{aligned} \tag{54}$$

For the purpose of renormalization, this lengthy expression for $\widehat{\Sigma}_2$ is not in the most suitable form . For the following counterterm Lagrangian, powers of the spin matrix are reduced with

the relations

$$\{S^\mu, S^\nu\} = \frac{1}{2}(v^\mu v^\nu - g^{\mu\nu}), \qquad [S^\mu, S^\nu] = i\varepsilon^{\mu\nu\rho\sigma} v_\rho S_\sigma. \tag{55}$$

In addition to various $SU(2)$ identities, I have also used the curvature relation

$$\Gamma_{\mu\nu} = \frac{1}{4}[u_\mu, u_\nu] - \frac{i}{2} f_{+\mu\nu}. \tag{56}$$

**6.** With the conventions of Ref. [7] for separating the finite part in dimensional regularization, we decompose $Z_{\rm irr}[j,\rho_v]$ in (26) into a finite and a divergent part, both depending on the arbitrary scale parameter $\mu$:

$$\begin{aligned} \Sigma_1(y,y')\delta^4(y-y') + \Sigma_2(y,y') &= \Sigma_1^{\rm fin}(y,y';\mu)\delta^4(y-y') + \Sigma_2^{\rm fin}(y,y';\mu) \\ &\quad - \frac{2\Lambda(\mu)}{F^2}\delta^4(y-y')[\widehat{\Sigma}_1(y) + \widehat{\Sigma}_2(y)] \\ \Lambda(\mu) &= \frac{\mu^{d-4}}{(4\pi)^2}\left\{\frac{1}{d-4} - \frac{1}{2}[\log 4\pi + 1 + \Gamma'(1)]\right\}. \end{aligned} \tag{57}$$

The generating functional $Z[j,\rho_v]$ can now be renormalized by introducing the counterterm Lagrangian

$$\mathcal{L}_{\rm ct}^{(3)}(x) = \frac{1}{(4\pi F)^2}\sum_i B_i \bar{N}_v(x) O_i(x) N_v(x) \tag{58}$$

with dimensionless coupling constants $B_i$ and field monomials $O_i(x)$ of $O(p^3)$. In analogy to Eq. (57), the low–energy constants $B_i$ are decomposed as

$$B_i = B_i^r(\mu) + (4\pi)^2 \beta_i \Lambda(\mu). \tag{59}$$

The $\beta_i$ are dimensionless functions of $g_A$ designed to cancel the divergences of the one–loop functional. In Table 1, the operators $O_i$ are listed together with the coefficients $\beta_i$.

This completes the renormalization of Green functions to $O(p^3)$. The sum of the irreducible one–loop functional $Z_{\rm irr}[j,\rho_v]$ and of the counterterm functional is finite and scale independent:

$$\begin{aligned} & Z_{\rm irr}[j,\rho_v] + \frac{1}{(4\pi F)^2}\sum_i B_i \int d^4x d^4x' d^4y \bar{\rho}_v(x) A_{(1)}^{-1}(x,y) O_i(y) A_{(1)}^{-1}(y,x')\rho_v(x') = \\ &= \int d^4x d^4x' d^4y d^4y' \bar{\rho}_v(x) A_{(1)}^{-1}(x,y)[\Sigma_1^{\rm fin}(y,y';\mu)\delta^4(y-y') + \Sigma_2^{\rm fin}(y,y';\mu) \\ &\qquad + \frac{1}{(4\pi F)^2}\delta^4(y-y')\sum_i B_i^r(\mu) O_i(y)] A_{(1)}^{-1}(y',x')\rho_v(x'). \end{aligned} \tag{60}$$

Since the same is true for the sum of the reducible contributions of Fig. 2, the total generating functional of $O(p^3)$ has been rendered finite and scale independent.

The renormalized low–energy constants $B_i^r(\mu)$ are measurable quantities satisfying the renormalization group equations

$$\mu\frac{d}{d\mu}B_i^r(\mu) = -\beta_i \tag{61}$$

implying

$$B_i^r(\mu) = B_i^r(\mu_0) - \beta_i \log\frac{\mu}{\mu_0}. \tag{62}$$

Table 1: Counterterms and their $\beta$–functions as defined in Eqs. (58, 59)

| i | $O_i$ | $\beta_i$ |
|---|---|---|
| 1 | $i[u_\mu, v\cdot\nabla u^\mu]$ | $g_A^4/8$ |
| 2 | $i[u_\mu, \nabla^\mu v\cdot u]$ | $-(1+5g_A^2)/12$ |
| 3 | $i[v\cdot u, v\cdot\nabla v\cdot u]$ | $(4-g_A^4)/8$ |
| 4 | $S\cdot u\langle u\cdot u\rangle$ | $g_A(4-g_A^4)/8$ |
| 5 | $u_\mu\langle u^\mu S\cdot u\rangle$ | $g_A(6-6g_A^2+g_A^4)/12$ |
| 6 | $S\cdot u\langle(v\cdot u)^2\rangle$ | $-g_A(8-g_A^4)/8$ |
| 7 | $v\cdot u\langle S\cdot uv\cdot u\rangle$ | $-g_A^5/12$ |
| 8 | $[\chi_-, v\cdot u]$ | $(1+5g_A^2)/24$ |
| 9 | $S\cdot u\langle\chi_+\rangle$ | $g_A(4-g_A^2)/8$ |
| 10 | $\nabla^\mu f_{+\mu\nu}v^\nu$ | $-(1+5g_A^2)/6$ |
| 11 | $iS^\mu v^\nu[f_{+\mu\nu}, v\cdot u]$ | $g_A$ |
| 12 | $iv_\lambda\varepsilon^{\lambda\mu\nu\rho}\langle u_\mu u_\nu u_\rho\rangle$ | $-g_A^3(4+3g_A^2)/16$ |
| 13 | $v_\lambda\varepsilon^{\lambda\mu\nu\rho}S_\rho\langle(v\cdot\nabla u_\mu)u_\nu\rangle$ | $-g_A^4/4$ |
| 14 | $v_\lambda\varepsilon^{\lambda\mu\nu\rho}\langle f_{+\mu\nu}u_\rho\rangle$ | $-g_A^3/4$ |
| 15 | $i(v\cdot\nabla)^3$ | $-3g_A^2$ |
| 16 | $v\cdot\overleftarrow{\nabla}\, S\cdot uv\cdot\nabla$ | $g_A^3$ |
| 17 | $\langle u\cdot u\rangle iv\cdot\nabla +$ h.c. | $-3g_A^2(4+3g_A^2)/16$ |
| 18 | $\langle(v\cdot u)^2\rangle iv\cdot\nabla +$ h.c. | $(8+9g_A^4)/16$ |
| 19 | $(v\cdot\nabla S\cdot u)v\cdot\nabla +$ h.c. | $g_A^3/3$ |
| 20 | $\langle\chi_+\rangle iv\cdot\nabla +$ h.c. | $-9g_A^2/16$ |
| 21 | $v_\lambda\varepsilon^{\lambda\mu\nu\rho}S_\rho u_\mu u_\nu v\cdot\nabla +$ h.c. | $-g_A^2(4+g_A^2)/4$ |
| 22 | $iv_\lambda\varepsilon^{\lambda\mu\nu\rho}S_\rho f_{+\mu\nu}v\cdot\nabla +$ h.c. | $g_A^2$ |

The operators in Table 1 constitute a minimal set for renormalizing the irreducible self–energy functional for off–shell nucleons. As long as one is interested in Green functions with on–shell nucleons only, the number of operators in (58) can be further reduced by invoking the equations of motion for the nucleons. We shall not pursue this option here as it requires some additional discussion.

Partial results for one–loop divergences in chiral $SU(2)$ have been obtained before. In particular, the results in Table 1 agree with those of Ref. [10]. Some of the coefficients have also been calculated recently by Park, Min and Rho: the coefficients $c_i$ in Eq. (43) of their paper [6] agree with Table 1, except for $c_4$ which should be a factor two smaller.

**7.** In addition to phenomenological applications for chiral $SU(2)$, the methods of this paper can be applied and extended in various ways:

i) Chiral $SU(3)$.

ii) Inclusion of higher baryon states, especially in view of recent results on the $1/N_C$ expansion for baryons (see Ref. [15] and references therein).

iii) Renormalization of the one–loop functional of $O(p^4)$.

iv) Non–leptonic weak interactions of baryons.

v) Combination of heavy quark and chiral symmetries.

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## Figure Captions

**Fig. 1:** Irreducible one–loop diagrams. The full (dashed) lines denote the nucleon (meson) propagators. The double lines indicate that the propagators (as well as the vertices) have the full tree–level structure attached to them as functionals of the external fields.

**Fig. 2:** Reducible diagrams of $O(p^3)$. The cross denotes counterterms from the meson Lagrangian $\mathcal{L}_4$ of $O(p^4)$.

## References


[1] J. Gasser, M.E. Sainio and A. Švarc, Nucl. Phys. B307 (1988) 779.

[2] E. Jenkins and A.V. Manohar, Phys. Lett. B255 (1991) 558.

[3] N. Isgur and M. Wise, Phys. Lett. B232 (1989) 113; ibid. B237 (1990) 527;
B. Grinstein, Nucl. Phys. B339 (1990) 253;
E. Eichten and B. Hill, Phys. Lett. B234 (1990) 511;
H. Georgi, Phys. Lett. B240 (1990) 447.

[4] E. Jenkins and A.V. Manohar, in Proc. of the Workshop on Effective Field Theories of the Standard Model, Dobogókö, Hungary, Aug. 1991, Ed. U.-G. Meißner, World Scientific (Singapore, 1992).

[5] U.-G. Meißner, Rep. Prog. Phys. 56 (1993) 903.

[6] T.-S. Park, D.-P. Min and M. Rho, Phys. Reports 233 (1993) 341.

[7] J. Gasser and H. Leutwyler, Ann. Phys. 158 (1984) 142.

[8] S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177 (1969) 2239;
C. Callan, S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177 (1969) 2247.

[9] T. Mannel, W. Roberts and Z. Ryzak, Nucl. Phys. B368 (1992) 204.

[10] V. Bernard, N. Kaiser, J. Kambor and U.-G. Meißner, Nucl. Phys. B388 (1992) 315.

[11] G. Ecker, The standard model at low energies, to appear in Proc. of the $6^{\text{th}}$ Indian–Summer School on Intermediate Energy Physics, Prague, Aug. 1993, preprint UWThPh-1993-31, hep-ph/9309268.

[12] M. Luke and A.V. Manohar, Phys. Lett. B286 (1992) 348.

[13] R.D. Ball, Phys. Reports 182 (1989) 1.

[14] I. Jack and H. Osborne, Nucl. Phys. B207 (1982) 474.

[15] R. Dashen, E. Jenkins and A.V. Manohar, The $1/N_C$ expansion for baryons, preprint UCSD/PTH 93-21, hep-ph/9310379.